2.4 ⋆⋆ **(a)** In a short time dt the projectile moves a distance vdt, and the front sweeps out a cylinder of volume $Avdt$. Therefore the mass of fluid encountered is $\varrho Avdt$, and the rate at which mass is swept up is ϱAv.

(b) If a mass $\varrho Avdt$ is accelerated from 0 to v in time dt, the rate of change of its momentum is ϱAv^2. This is, therefore, the forward force on the fluid and, hence, the backward force on the projectile.

(c) Since $A \propto D^2$, it follows that $f_{\text{quad}} = \kappa\varrho Av^2 = cv^2$, where $c = \kappa\varrho A \propto D^2$. For a sphere in air, $\kappa = 1/4$, $A = \pi D^2/4$, and $\varrho = 1.29\ \text{kg/m}^3$, so $f_{\text{quad}} = (\kappa\varrho\pi D^2/4)v^2 = cv^2$, where $c = \gamma D^2$ and

$$\gamma = \kappa\varrho\pi/4 = \tfrac{1}{4} \times (1.29\ \text{kg/m}^3) \times \pi/4 = 0.25\ \text{N}\cdot\text{s}^2/\text{m}^4.$$

2.6 ⋆ **(a)** If we insert the Taylor series for $e^{-t/\tau}$ into (2.33), we get

$$v_y(t) = v_{\text{ter}}\left[1 - e^{-t/\tau}\right] = v_{\text{ter}}\left[1 - \left(1 - \frac{t}{\tau} + \frac{t^2}{2\tau^2} - \cdots\right)\right].$$

The first two terms on the right cancel, and, if t is sufficiently small, we can neglect terms in t^2 and higher. This leaves us with

$$v_y(t) \approx v_{\text{ter}}t/\tau = gt$$

where to get the second equality I replaced v_{ter} by $g\tau$ as in (2.34).

(b) Putting $v_{yo} = 0$ into (2.35) and then inserting the Taylor series for the exponential, we find:

$$y(t) = v_{\text{ter}}t - v_{\text{ter}}\tau\left[1 - e^{-t/\tau}\right] = v_{\text{ter}}t - v_{\text{ter}}\tau\left[1 - \left(1 - \frac{t}{\tau} + \frac{t^2}{2\tau^2} - \cdots\right)\right].$$

On the right side, the second and third terms cancel, as do the first and fourth. If we neglect all terms beyond t^2, this leaves us with $y(t) \approx v_{\text{ter}}t^2/(2\tau) = \frac{1}{2}gt^2$, since $v_{\text{ter}} = g\tau$.

2.12 ⋆⋆ By the chain rule,

$$\dot{v} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} = \frac{1}{2}\frac{d(v^2)}{dx}.$$

This lets us rewrite the second law, $m\dot{v} = F$, as

$$\frac{d}{dx}(v^2) = \frac{2}{m}F(x),$$

which can be integrated to give

$$v^2 - v_o^2 = \frac{2}{m}\int_{x_o}^{x} F(x')dx'$$

as claimed. If F is constant, this reduces to the well-known kinematic result $v^2 - v_o^2 = 2a\,\Delta x$, where $a = F/m$ is the constant acceleration and $\Delta x = x - x_o$.

2.13 ⋆⋆ With $F = -kx$ and $v_o = 0$, Eq.(2.85) becomes

$$v^2 = -\frac{2k}{m}\int_{x_o}^{x} x'dx' = \omega^2({x_o}^2 - x^2) \quad \text{or} \quad v = -\omega\sqrt{{x_o}^2 - x^2} \tag{i}$$

where I have introduced the shorthand $\omega^2 = k/m$. [The second result is the square root of the first. Getting the right sign for the square root takes a little thought. Initially the velocity is clearly negative, and this is the phase of the motion I shall consider. After a while, the sign of v changes and the minus sign in (i) must be changed to a plus. Quite surprisingly, the final result is the same either way.]

Writing $v = dx/dt$ in (i), rearranging, and integrating, we find that

$$\omega t = -\int_{x_o}^{x} dx'/\sqrt{x_o^2 - x^2} = \arccos(x/x_o) \quad \text{or} \quad x = x_o\cos(\omega t),$$

which is simple harmonic motion. (To do the integral, I used the substitutions $x/x_o = u$ and then $u = \cos\theta$.)

2.18 ⋆ **(a)** If $f(x) = \ln(x)$, then, as you can easily check, $f(1) = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 2$, and $f^{(n)}(1) = (-1)^{n-1}(n-1)!$, so

$$\ln(1+\delta) = \delta - \frac{\delta^2}{2} + \frac{\delta^3}{3} - \frac{\delta^4}{4} + \cdots.$$

(b) If $f(x) = \cos(x)$, then $f(0) = 1$, $f'(0) = -\sin(0) = 0$, $f''(0) = -\cos(0) = -1$, $f'''(0) = \sin(0) = 0$, and so on. Thus

$$\cos(\delta) = 1 - \frac{\delta^2}{2!} + \frac{\delta^4}{4!} + \cdots.$$

(c) Similarly, $\sin(\delta) = \delta - \frac{\delta^3}{3!} + \frac{\delta^5}{5!} + \cdots$ and **(d)** $e^\delta = 1 + \delta + \frac{\delta^2}{2!} + \frac{\delta^3}{3!} + \cdots$.

2.19 $\star$ **(a)** In the absence of air resistance, we know that $x = v_{xo}t$ and $y = v_{yo}t - \frac{1}{2}gt^2$. If we solve the first of these to give $t = x/v_{xo}$ and then substitute into the second, we find

$$y = \frac{v_{yo}}{v_{xo}}x - \tfrac{1}{2}g\left(\frac{x}{v_{xo}}\right)^2,$$

which is the equation of a parabola.

(b) As air resistance is switched off, $\tau \to \infty$, and the second term inside the log term of (2.37) becomes small. Thus we can use the Taylor series (2.40) for the log,

$$\ln\left(1 - \frac{x}{v_{xo}\tau}\right) = -\frac{x}{v_{xo}\tau} - \frac{1}{2}\left(\frac{x}{v_{xo}\tau}\right)^2 - \cdots,$$

in (2.37). For τ sufficiently large, we can neglect all remaining terms in this series and (2.37) becomes

$$y \approx \frac{v_{yo} + v_{ter}}{v_{xo}}x - v_{ter}\tau\left(\frac{x}{v_{xo}\tau} + \frac{1}{2}\frac{x^2}{v_{xo}^2\tau^2}\right).$$

The second and third terms on the right cancel, and, if we replace v_{ter} by $g\tau$, the two remaining terms give precisely the answer to part (a).

2.23 $\star$ According to Eq.(2.59), $v_{ter} = \sqrt{mg/\gamma D^2}$. Since $m = \frac{4}{3}\pi R^3\varrho = \frac{1}{6}\pi D^3\varrho$, we can eliminate either m or D to give

$$v_{ter} = \sqrt{\frac{\pi D\varrho g}{6\gamma}} = \left(\frac{\pi\varrho}{6m}\right)^{1/3}\sqrt{\frac{mg}{\gamma}}. \qquad \text{(ii)}$$

In all three cases, $g = 9.8$ m/s^2 and $\gamma = 0.25$ kg/m^3.

(a) With $D = 3$ mm and $\varrho = 8$ g/cm^3, the second expression in Eq.(ii) gives $v_{ter} = 22$ m/s.

(b) With $m = 16 \times 0.454 = 7.26$ kg and $\varrho = 8$ g/cm^3, the third expression in Eq.(ii) gives $v_{ter} = 140$ m/s.

(c) With $m = 200 \times 0.454 = 90.8$ kg and $\varrho = 1$ g/cm^3, the third expression in Eq.(ii) gives $v_{ter} = 107$ m/s.

2.27 ⋆ If we choose our x axis pointing straight up the slope, then for the upward journey the x component of the second law reads

$$m\dot{v} = -cv^2 - mg\sin\theta = -c(v^2 + v_{\rm ter}^{\ 2})$$

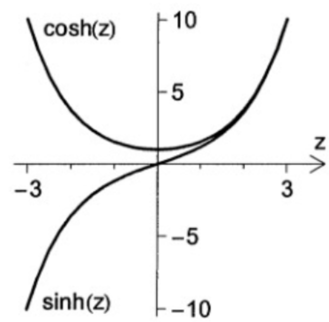
where v denotes the x component of the velocity and I have introduced the terminal speed for the puck on the incline, defined so that $v_{\rm ter}^{\ 2} = (mg\sin\theta)/c$. If we write this in the separated form $m\,dv/(v^2 + v_{\rm ter}^{\ 2}) = -c\,dt$, we can integrate both sides (the left side from $v_{\rm o}$ to v and the right from 0 to t) to give

$$\frac{m}{v_{\rm ter}}[\arctan(v/v_{\rm ter}) - \arctan(v_{\rm o}/v_{\rm ter})] = -ct \qquad \text{(iii)}$$

which can be solved to give

$$v = v_{\rm ter}\tan\bigl(\arctan(v_{\rm o}/v_{\rm ter}) - cv_{\rm ter}t/m\bigr).$$

Putting $v = 0$ in Eq.(iii), we find that the time to reach the top is $t = (m/cv_{\rm ter})\arctan(v_{\rm o}/v_{\rm ter})$.



2.33 ⋆⋆ **(a)** Note that when z is large and positive,

$$\cosh z \approx \sinh z \approx e^z/2.$$

Similarly, when z is large and negative,

$$\cosh z \approx -\sinh z \approx e^{-z}/2.$$

Also

$$\cosh(0) = 1 \quad \text{and} \quad \sinh(0) = 0.$$

(b) $\cos(iz) = \dfrac{e^{i(iz)} + e^{-i(iz)}}{2} = \dfrac{e^{-z} + e^{z}}{2} = \cosh(z).$

Similarly, $\sinh(z) = -i\sin(iz)$.

(c) $\dfrac{d}{dz}\cosh(z) = \dfrac{d}{dz}\dfrac{e^z + e^{-z}}{2} = \dfrac{e^z - e^{-z}}{2} = \sinh(z)$, and likewise $\dfrac{d}{dz}\sinh(z) = \cosh(z)$.

Integrating these two results, we find that

$$\int \sinh(z)dz = \cosh(z) \qquad \text{and} \qquad \int \cosh(z)dz = \sinh(z)$$

(d) $\cosh^2(z) - \sinh^2(z) = [\cos(iz)]^2 - [-i\sin(iz)]^2 = [\cos(iz)]^2 + [\sin(iz)]^2 = 1$

(e) If we make the substitution $x = \sinh(z)$, then

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\cosh z\,dz}{\sqrt{1+\sinh^2 z}} = \int dz = z = \text{arcsinh}(x).$$

2.39 ⋆⋆ **(a)** The equation of motion is $m\dot{v} = -f_{\text{fr}} - cv^2$, which separates to give

$$-m\frac{dv}{f_{\text{fr}} + cv^2} = dt.$$

This can be integrated from time 0 to t (and velocity from v_{o} to v). The integral over v gives an arctan function. (Make the substitutions $cv^2/f_{\text{fr}} = u^2$ and then $u = \tan w$.) The result is

$$t = \frac{m}{\sqrt{f_{\text{fr}}c}}\left(\arctan\sqrt{\frac{c}{f_{\text{fr}}}}v_{\text{o}} - \arctan\sqrt{\frac{c}{f_{\text{fr}}}}v\right).$$

(b) Putting in the numbers, with $v_{\text{o}} = 20$ m/s and the four given final velocities $v = 15, 10, 5,$ and 0 m/s, we find the following corresponding times:

v (m/s)	15	10	5	0
t (s)	6.3	18.4	48.3	142

The corresponding times if we neglect friction are (from Problem 2.26) 6.7, 20.0, 60.0, and ∞. To neglect friction, compared to the quadratic air resistance, is quite good at higher speeds, but terrible at very low speeds.

2.47 ⋆ **(a)** $z = 6 + 8i = 10e^{i\theta}$ and $w = 3 - 4i = 5e^{-i\theta}$, where $\theta = 0.927$ rad. (Note that the phase angles of z and w are exactly opposite — same θ in both expresions.) Therefore

$$z + w = 9 + 4i \quad \text{and} \quad z - w = 3 + 12i,$$

$$zw = (10e^{i\theta})(5e^{-i\theta}) = 50,$$

and

$$\frac{z}{w} = \frac{10e^{i\theta}}{5e^{-i\theta}} = 2e^{2i\theta} = 2\cos(2\theta) + 2i\sin(2\theta) = -0.56 + 1.92i,$$

or

$$\frac{z}{w} = \frac{zw^*}{ww^*} = \frac{(6+8i)(3+4i)}{(3-4i)(3+4i)} = \frac{-14+48i}{25} = -0.56 + 1.92i.$$

(b) $z = 8e^{i\pi/3} = 4 + 4\sqrt{3}i$ and $w = 4e^{i\pi/6} = 2\sqrt{3} + 2i$. Therefore,

$$z + w = (4 + 2\sqrt{3}) + (4\sqrt{3} + 2)i \quad \text{and} \quad z - w = (4 - 2\sqrt{3}) + (4\sqrt{3} - 2)i,$$

$$zw = (8e^{i\pi/3})(4e^{i\pi/6}) = 32e^{i\pi/2} = 32i \quad \text{and} \quad \frac{z}{w} = \frac{8e^{i\pi/3}}{4e^{i\pi/6}} = 2e^{i\pi/6} = \sqrt{3} + i.$$

2.53 ⋆ The components of the force are $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = q(v_y B, -v_x B, E)$, so the three components of $m\mathbf{a} = \mathbf{F}$ are

$$m\dot{v}_x = qBv_y, \qquad m\dot{v}_y = -qBv_x, \qquad m\dot{v}_z = qE.$$

The first two of these are exactly the same as (2.64) and (2.65) for the case of no electric field, and the motion of x and y is therefore the same as in Figure 2.15: The transverse position (x, y) moves clockwise around a circle at constant angular velocity $\omega = qB/m$. The equation for v_z shows that there is a constant acceleration in the z direction, $a_z = qE/m$, so that $z = z_o + v_{zo}t + \frac{1}{2}a_z t^2$. The particle moves in a helix or spiral of constant radius around a line parallel to the z axis, with an increasing pitch as the motion in the z direction accelerates.